Journal of Approximation Theory **94**, 481–493 (1998)
Article No. AT983189

Approximation of k-Monotone Functions

Kirill A. Kopotun

Department of Mathematics, Vanderbilt University, Nashville, Tennessee 37240

Communicated by Manfred v. Golitschek

Received December 23, 1996; accepted June 4, 1997

It is shown that an algebraic polynomial of degree $\leqslant k-1$ which interpolates a k-monotone function f at k points, sufficiently approximates it, even if the points of interpolation are close to each other. It is well known that this result is not true in general for non-k-monotone functions. As an application, we prove a (positive) result on simultaneous approximation of a k-monotone function and its derivatives in $\mathbf{L}_p$, $0<p<1$, metric, and also show that the rate of the best algebraic approximation of k-monotone functions (with bounded $(k-2)$nd derivatives in $\mathbf{L}_p$, $1<p<\infty$, is $o(n^{-k/p})$.

1. INTRODUCTION AND MAIN RESULTS

It is well known that a polynomial of degree $\leqslant k-1$ which interpolates a continuous function f at k points, which are not too close to each other, sufficiently approximates it. For example, one of the interpolatory versions of the well known Whitney theorem (in $\mathbf{C}$ norm) states:

> Let $f \in \mathbf{C}[a, b]$, and let p_{k-1} be a polynomial of degree $\leqslant k-1$ which interpolates f at $a+\nu(b-a)/(k-1)$, $\nu=0, 1, ..., k-1$. Then $\|f-p_{k-1}\|_{\mathbf{C}[a, b]} \leqslant C\omega^k(f, b-a, [a, b])_\infty$.

Here, ω^k is the usual kth modulus of smoothness which is defined by

$$\omega^k(f, t, [a, b])_p := \sup_{0<h\leqslant t} \|\Delta_h^k(f, \cdot, [a, b])\|_{\mathbf{L}_p[a, b]}, \qquad 0<p\leqslant\infty,$$

where

$$\Delta_h^k(f, x, [a, b]) := \begin{cases} \sum_{i=0}^{k} \binom{k}{i} (-1)^{k-i} f\left(x-\frac{kh}{2}+ih\right), & \text{if } \; x \pm \frac{kh}{2} \in [a, b], \\ 0, & \text{otherwise,} \end{cases}$$

0021-9045/98 $25.00

is the kth symmetric difference.

It is not difficult to see that the condition that the points of interpolation are not too close to each other cannot be removed. Indeed, let

$$f(x)=\begin{cases} x, & -1\leqslant x<0,\\ 0, & 0\leqslant x<\varepsilon,\\ x-\varepsilon, & \varepsilon\leqslant x\leqslant 1.\end{cases}$$

Then $\omega^2(f, 2, [-1, 1])_\infty=\omega^2(f-x, 2, [-1, 1])_\infty\leqslant 4\,\|f-x\|_{\mathbf{C}[-1,1]}\leqslant 4\varepsilon$. At the same time, let p_1 interpolate f at $x=0$ and $x=\varepsilon$ (i.e., $p_1(x)=0$ for all x). Then the estimate

$$\|f-p_1\|_{\mathbf{C}[-1,1]}\leqslant C\omega^2(f, 2, [-1, 1])_\infty$$

does not hold, since, otherwise

$$1=\|f\|_{\mathbf{C}[-1,1]}=\|f-p_1\|_{\mathbf{C}[-1,1]}\leqslant C\omega^2(f, 2, [-1, 1])_\infty\leqslant C\varepsilon,$$

which, of course, is not true if ε is small.

It turns out that the situation is different if f is convex or, more generally, k-monotone, even if approximation in $\mathbf{L}_p$, $0<p\leqslant\infty$, metric, is considered. A function $f: [a, b]\mapsto\mathscr{R}$ is said to be k-monotone, $k\geqslant 1$, on $[a, b]$ iff for all choices of $(k+1)$ distinct $x_0, ..., x_k$ in $[a, b]$ the inequality

$$[x_0, ..., x_k]\, f\geqslant 0 \tag{1}$$

holds, where $[x_0, ..., x_k]\, f=\sum_{j=0}^{k}(f(x_j)/w'(x_j))$ denotes the kth divided difference of f at $x_0, ..., x_k$, and $w(x)=\prod_{j=0}^{k}(x-x_j)$. Note that 1-monotone (2-monotone) functions are just nondecreasing (convex) functions. The class of all k-monotone functions on $[a, b]$ (or (a, b)) is denoted by $\Delta^k[a, b]$ (or $\Delta^k(a, b)$). If $f\in\mathbf{C}^k[-1, 1]$, then $f\in\Delta^k[-1, 1]$ iff $f^{(k)}(x)\geqslant 0$, $x\in[-1, 1]$.

A function f is called *weakly k-monotone* if (1) is satisfied for any set of equally spaced points $x_0, ..., x_k$. In general, weakly k-monotone functions do not have to be k-monotone. In fact, they do not even have to be continuous. However, if f is *assumed* to be continuous, then the concepts of k-monotonicity and weak k-monotonicity turn out to be equivalent. We refer the reader to the book of Roberts and Varberg [10] for further discussions, and just mention the following result which is due to Ciesielski [2]:

> Let f be weakly k-monotone on $[a, b]$ for some $k\geqslant 2$. If f is bounded on at least one subset E of (a, b) having positive measure, then f is continuous on (a, b).

The following theorem about differentiability of k-monotone functions (see [10, 9], for example) will be useful.

THEOREM A. *Suppose for some* $k \geqslant 2$ *that* $f: [a, b] \mapsto \mathscr{R}$ *is* k*-monotone. Then* $f^{(j)}(x)$, *the derivative of order* j, *exists on* (a, b) *for* $j \leqslant k-2$ *and is* $(k-j)$*-monotone. In particular,* $f^{(k-2)}(x)$ *exists, is convex, and therefore satisfies a Lipschitz condition on any closed interval* $[\xi, \zeta]$ *contained in* (a, b), *is absolutely continuous on* $[\xi, \zeta]$, *is continuous on* (a, b), *and has left and right (nondecreasing) derivatives,* $f_-^{(k-1)}(x)$ *and* $f_+^{(k-1)}(x)$ *on* (a, b). *Moreover, the set* E *where* $f^{(k-1)}(x)$ *fails to exist is countable, and* $f^{(k-1)}$ *is continuous on* $(a, b)\backslash E$.

We now introduce some notations that are used throughout this paper. A function f is said to change sign at y if $f(x) \leqslant 0$ $(\geqslant 0)$ for $x < y$ and $f(x) \geqslant 0$ $(\leqslant 0)$ for $x > y$. (Note that, with this definition, the zero function changes sign at every point.)

An interval $[a, b]$ is often denoted by J, in which case $|J| = b - a$ and

$$[a - v\,|J|, b + v\,|J|] =: [(2v+1)\,J].$$

Also, let

$$\omega_\varphi^m(f, \delta)_p := \sup_{0 < h \leqslant \delta} \|\Delta_{h\varphi(\cdot)}^m(f, \cdot, [-1, 1])\|_p$$

denote the mth order Ditzian–Totik modulus of smoothness, and let

$$\bar{\Delta}_h^m(f, x, [a, b]) := \Delta_h^m(f, x + kh/2, [a, b])$$

be the forward mth difference. If the interval $[-1, 1]$ is used in any of the above notations, it will be omitted, for example, $\|f\|_p := \|f\|_{\mathbf{L}_p[-1,1]}$, $\omega^k(f, t)_p := \omega^k(f, t, [-1, 1])_p$, etc.

Finally, we emphasize that $\Delta^k := \Delta^k(-1, 1)$ and, thus, functions from Δ^k do not have to be defined at ± 1 (and, hence, they do not have to be bounded on $(-1, 1)$). For example, $(x+1)^{-1} \in \Delta^2$. At the same time, if $f \in \Delta^k[-1, 1]$, then it is bounded on $[-1, 1]$. Moreover, since $f \in \mathbf{C}(-1, 1)$ (if $k \geqslant 2$) and f is nondecreasing (if $k = 1$), then $f \in \mathbf{L}_p[-1, 1]$ for all $0 < p \leqslant \infty$.

The following theorems are the main results of this paper.

THEOREM 1. *Let* $f \in \Delta^k$, $k \geqslant 1$, *and* $J = [a, b]$ *be such that*

$$\text{(i)}\quad [3J] \subset [-1, 1] \qquad \textit{or} \qquad \text{(ii)}\quad [(4k+1)\,J] \subset [-1, 1],$$

and let p_{k-1} be a polynomial of degree $\leqslant k-1$ which interpolates f at k points in J. If $f^{(m)} \in \mathbf{L}_p[-1, 1]$, $0 < p \leqslant \infty$, for some m such that $0 \leqslant m \leqslant k-1$, then the following inequalities hold for all $j = 0, 1, ..., m$:

$$\text{(i)} \quad \|f^{(j)} - p_{k-1}^{(j)}\|_{\mathbf{L}_p[b,\, b+|J|]} \leqslant C\omega^{k-j}(f^{(j)}, |J|, [b - 2k\,|J|, b + |J|])_p \quad \text{if} \quad b - 2k\,|J| \geqslant -1, \tag{2}$$

$$\text{(i)} \quad \|f^{(j)} - p_{k-1}^{(j)}\|_{\mathbf{L}_p[a-|J|,\, a]} \leqslant C\omega^{k-j}(f^{(j)}, |J|, [a - |J|, a + 2k\,|J|])_p \quad \text{if} \quad a + 2k\,|J| \leqslant 1, \tag{3}$$

or

$$\text{(ii)} \quad \|f^{(j)} - p_{k-1}^{(j)}\|_{\mathbf{L}_p[3J]} \leqslant C\omega^{k-j}(f^{(j)}, |J|, [(4k+1)\,J])_p. \tag{4}$$

THEOREM 2. *Let $f \in \Delta^k$, $k \geqslant 1$, be such that $f^{(m)} \in \mathbf{L}_p[-1, 1]$, $0 < p \leqslant \infty$, for some m, $0 \leqslant m \leqslant k-1$. Then there exists $p_n \in \mathbf{P}_n$ such that*

$$\|f^{(j)} - p_n^{(j)}\|_p \leqslant C\omega_\varphi^{k-j}(f^{(j)}, n^{-1})_p \tag{5}$$

for all $j = 0, 1, ..., m$, where C depends only on k and p (if $p < 1$), and the construction of the polynomial p_n does not depend on p. Moreover, these estimates are exact in the sense that

$$\|f - p_n\|_p \leqslant C\omega^{k+1}(f, n^{-1})_p$$

and

$$\|f' - p_n'\|_p \leqslant C\omega^k(f', n^{-1})_p$$

cannot hold simultaneously for $f \in \Delta^k$ if $0 < p < 1$.

Note, that if $1 \leqslant p \leqslant \infty$, then better estimates than (5) are valid (moreover these estimates are true for general functions f which do not have to be in Δ^k). See [5, 6] for more details.

THEOREM 3. *Let $f \in \Delta^k[-1, 1]$ be such that $f^{(k-2)} \in \mathbf{C}[-1, 1]$. Then there exists a polynomial $p_n \in \mathbf{P}_n$ such that*

$$\|f - p_n\|_p \leqslant Cn^{-k/p}\omega_\varphi^k(f, n^{-1})_\infty^{1/q}\, \|f^{(k-2)}\|_\infty^{1/p}, \tag{6}$$

for all $1 \leqslant p \leqslant \infty$, where $1/p + 1/q = 1$, and C depends only on k. In particular,

$$E_n(f)_p = o(n^{-k/p}), \qquad 1 < p < \infty,$$

where $E_n(f)_p = \inf_{p_n \in \mathbf{P}_n} \|f - p_n\|_p$.

Theorem 3 generalizes the results of Ivanov [4] and Stojanova [11] (see also [7, 8]), which were proved for convex functions.

2. LOCAL APPROXIMATION OF k-MONOTONE FUNCTIONS BY INTERPOLATORY POLYNOMIALS

THEOREM 4. *Let $J := [a, b]$ be such that $[3J] \subset [-1, 1]$, and let $f \in \Delta^k$, $k \geqslant 1$, be such that $f(y_1) = \cdots = f(y_k) = 0$, where $a < y_1 < \cdots < y_k < b$. If $f \in \mathbf{L}_p[-1, 1]$, $0 < p \leqslant \infty$, then*

$$\|f\|_{\mathbf{L}_p[b,\, b+|J|]} \leqslant C\omega^k(f, 2\,|J|, [b-2k\,|J|, b+|J|])_p \qquad \text{if} \quad b-2k\,|J| \geqslant -1, \tag{7}$$

and

$$\|f\|_{\mathbf{L}_p[a-|J|,\, a]} \leqslant C\omega^k(f, 2\,|J|, [a-|J|, a+2k\,|J|])_p \qquad \text{if} \quad a+2k\,|J| \leqslant 1, \tag{8}$$

where constants C depend only on k and p (if $0 < p < 1$), and do not depend on the (location of) interpolation points.

(We want to emphasize once more that $f \in \Delta^k$ does not have to be bounded (near the endpoints) and, thus, the condition $f \in \mathbf{L}_p[-1, 1]$ is needed in the statement (of course, if we require that $f \in \Delta^k$ and f is *defined* at ± 1, then $f \in \mathbf{C}(-1, 1)$ (if $k \geqslant 2$) and is bounded on $[-1, 1]$, and this condition can be removed.)

LEMMA B [1]. *Let $f \in \Delta^k$, $k \geqslant 1$, and let $l_{k-1}(x)$ interpolate f at $z_1, ..., z_k$. Then $f - l_{k-1}$ changes sign at $z_1, ..., z_k$ and, in particular, $f(x) - l_{k-1}(x) \geqslant 0$ for $x \geqslant \max\{z_1, ..., z_k\}$.*

At this stage we note that this lemma (as well as almost all results below) can be slightly generalized in the case $k = 1$. Indeed, $f \in \Delta^1$ does not have to be continuous on $(-1, 1)$ and, therefore, the requirement that $L_0(x)$ is an interpolatory Lagrange polynomial is too restrictive. Suppose that f is discontinuous at $z \in (-1, 1)$. Then the assertion of the lemma is still valid for $L_0(x) := \alpha$, $x \in [-1, 1]$, if $\lim_{x \to z^-} f(x) \leqslant \alpha \leqslant \lim_{x \to z^+} f(x)$. To simplify the exposition we do not mention or discuss this later in the paper.

LEMMA 5. *Let $f \in \Delta^k$, $k \geqslant 1$, be such that $f(y_1) = \cdots = f(y_k) = 0$, where $-1 < y_1 < \cdots < y_k < 1$, and let $L_{k-1}(x) := L_{k-1}(x, f; t_1, t_2, ..., t_k)$ be the*

Lagrange polynomial interpolating f *at* $t_1, ..., t_k$. *If* $-1 < t_1 < t_2 < \cdots < t_k < y_1$, *then* $L_{k-1}(x) \leqslant 0$ *for* $x > y_k$.

Proof. We will give a proof by induction on k. Let $k=1$, $f \in \Delta^1$, $f(y_1)=0$, and $L_0(x) := L_0(x, f; t_1)$, where $t_1 < y_1$. Then $L_0(x) \leqslant 0$ for all x (since $f(t_1) \leqslant 0$). Let $k=2$, $f \in \Delta^2$, $f(y_1) = f(y_2) = 0$ for $y_1 < y_2$, and $L_1(x) = L_1(x, f; t_1, t_2)$, where $t_1 < t_2 < y_1$. Then $L_1(x)$ is decreasing, and, by Lemma B, $L_1(y_1) \leqslant f(y_1) = 0$. Hence, $L_1(x) \leqslant 0$ for $x \geqslant y_1$ and, in particular, for $x > y_2$.

Suppose now that the lemma is proved for $k-1$ and that it is not valid for k ($k \geqslant 3$). Then, there exists $t > y_k$ such that $L_{k-1}(t) > 0$. At the same time, Lemma B implies that $L_{k-1}(x) \leqslant f(x)$ for $x \geqslant t_k$. In particular, $L_{k-1}(y_k) \leqslant f(y_k) = 0$. Therefore, there exists $\xi \in (y_k, t)$ such that $L'_{k-1}(\xi) = (L_{k-1}(t) - L_{k-1}(y_k))/(t - y_k) > 0$. Rolle's theorem implies that $L'_{k-1}(x)$ interpolates f' at $k-1$ points $\tilde{t}_1, ..., \tilde{t}_{k-1}$, where $t_1 < \tilde{t}_1 < t_2 < \cdots < t_{k-1} < \tilde{t}_{k-1} < t_k < y_1$. Also, $f' \in \Delta^{k-1}$ and $f'(\tilde{y}_1) = \cdots = f'(\tilde{y}_{k-1}) = 0$, where $y_1 < \tilde{y}_1 < y_2 < \cdots < y_{k-1} < \tilde{y}_{k-1} < y_k$. Now, the fact that $L'_{k-1}(\xi) > 0$ for $\xi > y_k > \tilde{y}_{k-1}$ contradicts the statement of the lemma for $k-1$. This completes the proof. ∎

Proof of Theorem 4. We will only prove (7), since the proof of (8) is analogous. Let $x \in [b, b+|J|]$ be fixed, and consider $L_{k-1}(y) := L_{k-1}(y, f; x-2k\,|J|, x-2(k-1)\,|J|, ..., x-2\,|J|)$. Then

$$\begin{aligned} 0 \leqslant f(x) &= f(x) + \bar{\Delta}^k_{2\,|J|}\, L_{k-1}(x-2k\,|J|) \\ &= \bar{\Delta}^k_{2\,|J|}\, f(x-2k\,|J|) + L_{k-1}(x) \\ &\leqslant \bar{\Delta}^k_{2\,|J|}\, f(x-2k\,|J|), \end{aligned}$$

since, by Lemma 5, $L_{k-1}(x) \leqslant 0$. Thus,

$$|f(x)| \leqslant |\bar{\Delta}^k_{2\,|J|}\, f(x-2k\,|J|)|$$

for all $x \in [b, b+|J|]$. Integrating over $[b, b+|J|]$ we get

$$\begin{aligned} &\int_b^{b+|J|} |f(x)|^p\, dx \\ &\qquad \leqslant \int_b^{b+|J|} |\bar{\Delta}^k_{2\,|J|}\, f(x-2k\,|J|)|^p\, dx \\ &\qquad = \int_{b-2k\,|J|}^{b-(2k-1)\,|J|} |\bar{\Delta}^k_{2\,|J|}\, f(y)|^p\, dy \leqslant \omega^k(f, 2\,|J|, [b-2k\,|J|, b+|J|])^p_p. \end{aligned}$$

This completes the proof of the theorem. ∎

COROLLARY 6. *Let $f \in \Delta^k$, $k \geqslant 1$, and let p_{k-1} be a polynomial of degree $\leqslant k-1$ which interpolates f at k points in $J=[a, b]$. If $[3J] \subset [-1, 1]$ and $f \in \mathbf{L}_p[-1, 1]$, $0<p\leqslant\infty$, then*

$$\|f-p_{k-1}\|_{\mathbf{L}_p[b,\, b+|J|]} \leqslant C\omega^k(f, |J|, [b-2k\,|J|, b+|J|])_p \qquad \text{if} \quad b-2k\,|J| \geqslant -1, \tag{9}$$

and

$$\|f-p_{k-1}\|_{\mathbf{L}_p[a-|J|,\, a]} \leqslant C\omega^k(f, |J|, [a-|J|, a+2k\,|J|])_p \qquad \text{if} \quad a+2k\,|J| \leqslant 1, \tag{10}$$

where constants C depend only on k and p (if $0<p<1$), and do not depend on the (location of) interpolation points.

We need the following lemma.

LEMMA 7. *Let $f \in \Delta^k$, $k \geqslant 1$, be such that $f(y_1)=\cdots=f(y_k)=0$, where $y_1<\cdots<y_k$. Then f is nondecreasing for $x \geqslant y_k$.*

Proof. The statement is obvious for $k=1$ and $k=2$, and for $k\geqslant 3$ it immediately follows from Lemma B and the fact that $f' \in \Delta^{k-1}$ and $f'(\tilde{y}_1)=\cdots=f'(\tilde{y}_{k-1})=0$, where $y_1<\tilde{y}_1<y_2<\cdots<y_{k-1}<\tilde{y}_{k-1}<y_k$. ∎

THEOREM 8. *Let $f \in \Delta^k$, $k \geqslant 1$, $J=[a, b]$ be such that $[(4k+1)\,J] \subset [-1, 1]$, and let $p_{k-1} \in \Pi_{k-1}$ interpolate f at k points in J. If $f \in \mathbf{L}_p[-1, 1]$, $0<p\leqslant\infty$, then*

$$\|f-p_{k-1}\|_{\mathbf{L}_p([3J])} \leqslant C\omega^k(f, |J|, [(4k+1)\,J])_p, \tag{11}$$

where the constant C depends only on k and p (if $p<1$), and does not depend on the (location of) interpolation points.

Proof. It follows from Corollary 6 that

$$\|f-p_{k-1}\|_{\mathbf{L}_p[b,\, b+|J|]} \leqslant C\omega^k(f, |J|, [b-2k\,|J|, b+|J|])_p$$

and

$$\|f-p_{k-1}\|_{\mathbf{L}_p[a-|J|,\, a]} \leqslant C\omega^k(f, |J|, [a-|J|, a+2k\,|J|])_p.$$

We will now show that

$$\|f-p_{k-1}\|_{\mathbf{L}_p[a,\, b]} \leqslant C\omega^k(f, |J|, [b-2k\,|J|, b+2k\,|J|])_p.$$

Indeed, let $g := f - p_{k-1}$. Then $g \in \Delta^k$, $k \geqslant 1$, $g(y_1) = \cdots g(y_k) = 0$, $y_1 < \cdots < y_k$ in $[a, b]$. Let q_{k-1} be a polynomial in Π_{k-1} interpolating g at $b + i/k\,|J|$, $i = 0, 1, ..., k-1$. Corollary 6 implies that

$$\begin{aligned}\|g - q_{k-1}\|_{\mathbf{L}_p[a,b]} &\leqslant C\omega^k(g, |J|, [a, b + 2k\,|J|])_p\\ &= C\omega^k(f, |J|, [a, b+2k\,|J|])_p.\end{aligned}$$

Let us now estimate $\|q_{k-1}\|_{\mathbf{L}_p[a,b]}$. Using Lemma 7 we have

$$\begin{aligned}&\|q_{k-1}\|^p_{\mathbf{L}_p[a,b]}\\ &\quad= \int_a^b |q_{k-1}(x)|^p\,dx\\ &\quad\leqslant \int_a^b \left|\sum_{j=0}^{k-1} \prod_{i\neq j} \frac{x - (b + i\,|J|/k)}{(j-i)\,|J|/k}\, g\left(b + \frac{j}{k}\,|J|\right)\right|^p dx\\ &\quad\leqslant C\,|J| \sum_{j=0}^{k-1} \left|g\left(b + \frac{j}{k}\,|J|\right)\right|^p\\ &\quad\leqslant C\,|J| \sum_{j=0}^{k-1} \left(\frac{k}{|J|} \int_{b + j\,|J|/k}^{b + (j+1)\,|J|/k} |g(y)|^p\,dy\right)\\ &\quad\leqslant C \int_b^{b+|J|} |g(y)|^p\,dy\\ &\quad= C\,\|f - p_{k-1}\|_{\mathbf{L}_p[b, b+|J|]} \leqslant C\omega^k(f, |J|, [b - 2k\,|J|, b + |J|])^p_p.\end{aligned}$$

From the above estimate we have

$$\begin{aligned}\|f - p_{k-1}\|^p_{\mathbf{L}_p[a,b]} = \|g\|^p_{\mathbf{L}_p[a,b]} &\leqslant C(\|g - q_{k-1}\|^p_{\mathbf{L}_p[a,b]} + \|q_{k-1}\|^p_{\mathbf{L}_p[a,b]})\\ &\leqslant C\omega^k(f, |J|, [b - 2k\,|J|, b + 2k\,|J|])^p_p.\end{aligned}$$

Hence,

$$\|f - p_{k-1}\|_{\mathbf{L}_p[3J]} \leqslant C\omega^k(f, |J|, [b - 2k\,|J|, b + 2k\,|J|])_p.$$

The proof is complete. ∎

Note that, in a sense, (11) is the best of what one can expect. More precisely, the estimate

$$\|f - p_{k-1}\|_{\mathbf{L}_p(J)} \leqslant C\omega^k(f, |J|, [(4k+1)\,J] \cap [-1, 1])_p, \tag{12}$$

for example, is no longer true—the interval $J = [a, b]$ should be "far" from the endpoints. Indeed, let $f(x) = (x + 1 - \varepsilon)_+$ and let $[a, b] = [-1, -1/5]$, for example. Then $f \in \Delta^2$ and $\omega^2(f, 1)_p = \omega^2(f - x - 1, 1)_p \leqslant C \|f - x - 1\|_p \leqslant C\varepsilon$. Let p_1 interpolate f at $x = -1$ and $x = -1 + \varepsilon$. Then $p_1(x) = 0$ for all x, and $\|f - p_1\|_p = \|f\|_p \sim \text{const}$. This shows that (12) cannot hold in general.

Proof of Theorem 1. The assertion of Theorem 1 follows from Corollary 6, Theorems 8 and A, and the fact that $p_{k-1}^{(j)}$ interpolates $f^{(j)}$ at $k - j$ points in J for $j \leqslant k-2$ (see the Remark after Lemma B concerning the case $j = k-1$). ∎

3. SIMULTANEOUS APPROXIMATION IN L_p, $p > 0$

It is well known that L_p, $0 < p < 1$, spaces are "pathological in nature." For example, they are not Banach spaces, there are no continuous linear functionals in L_p (except the zero functional), the inequality $\omega^m(f, \delta)_p \leqslant C\delta\omega^{m-1}(f', \delta)_p$ is not true if $0 < p < 1$, etc. Ditzian [3] proved that, in general, the rate of simultaneous approximation of a function and its derivatives is very bad if $0 < p < 1$:

> For $0 < p < 1$ and $f \in AC[0, 1]$ we cannot have $P_n \in \mathbf{P}_n$ such that $\|f - P_n\|_p \leqslant C\omega^2(f, n^{-1})_p$ and $\|f' - P_n'\|_p \leqslant C\omega(f', n^{-1})_p$ simultaneously with constants independent of f and n.

Theorem 2 improves this result showing that if f is assumed to be in Δ^k, then simultaneous approximation of f and its derivatives is possible for $p < 1$, but only to some degree. We mention that our proof of the negative part of Theorem 2 is based on the construction used in [3].

Proof of Theorem 2. Let $x_j := \cos(j\pi/n)$, $0 \leqslant j \leqslant n$, $\Delta_n(x) := n^{-1}\sqrt{1-x^2} + n^{-2}$, and let a spline $\mathscr{L}_n(x, f)$ be defined as follows: $\mathscr{L}_n(x, f) = L_{k-1}(x, f; x_{\alpha-1}, x_{\alpha-1} + h_\alpha, ..., x_{\alpha-1} + (k-1)\, h_\alpha)$, $x \in [-1, x_\alpha)$, where $h_\alpha = 10^{-1}\Delta_n(x_{\alpha-1})$ and $\alpha = n - 8k^2$, $\mathscr{L}_n(x, f) = L_{k-1}(x, f; x_j, x_{j-1}, ..., x_{j-k+1})$ if $x \in [x_j, x_{j-1})$ for $8k^2 \leqslant j \leqslant n - 8k^2$, $\mathscr{L}_n(x, f) = L_{k-1}(x, f; x_\beta - (k-1)\, h_\beta, ..., x_\beta - h_\beta, x_\beta)$ if $x \in [x_{\beta-1}, 1]$, where $h_\beta = 10^{-1}\Delta_n(x_\beta)$ and $\beta = 8k^2$.

We remark that this somewhat complicated construction is needed in order to be able to use (3) near the left endpoint, (2) near the right endpoint, and (4) in the middle of $[-1, 1]$. Now, using the same sequence of estimates as in the proof of Theorem 1 of [5] together with Theorem 1 it is possible to show that

$$\|f^{(j)} - \mathscr{L}_n^{(j)}(f)\|_p \leqslant C\omega_\varphi^{k-j}(f^{(j)}, n^{-1})_p. \tag{13}$$

Let $q_t(x, f)$ denote the restriction of $\mathscr{L}_n(x, f)$ to $[x_t, x_{t-1}]$, $1 \leqslant t \leqslant n$. Then

$$\mathscr{L}_n(x, f) = q_n(x, f) + \sum_{t=1}^{n-1} (q_t(x, f) - q_{t+1}(x, f)) \chi_t(x),$$

where $\chi_t(x) = 1$ if $x \geqslant x_t$, and $\chi_t(x) = 0$ otherwise. The polynomial

$$P_n(x, f) = q_n(x, f) + \sum_{t=1}^{n-1} (q_t(x, f) - q_{t+1}(x, f)) T_t(x),$$

where $T_t(x)$ is a polynomial of degree $\leqslant Cn$ (defined in [5]), satisfies

$$\|\mathscr{L}_n^{(j)}(f) - P_n^{(j)}(f)\|_p \leqslant C\omega_\varphi^{k-j}(f^{(j)}, n^{-1})_p. \tag{14}$$

The proof of (14) is almost word for word the same as the proof of Theorem 1 of [5]. The only difference is that Theorem 1 should be used instead of Lemma A of [5], and, in the case $0 < p < 1$, the inequality $\|\sum_i f_i\|_p^p \leqslant \sum_i \|f_i\|_p^p$ replaces the Minkovski inequality. We omit the details.

Now, a sketch of the proof of the negative part of the theorem will be given. We employ the idea and construction which was used by Ditzian in [3]. It is more convenient to consider the interval $[0, 1]$ instead of $[-1, 1]$.

Let $S(x, m; a, b) := \int_a^x (y-a)^m (b-y)^m \, dy (\int_a^b (y-a)^m (b-y)^m \, dy)^{-1}$ and

$$g(x) = \begin{cases} nS(x, k; ln^{-2}, ln^{-2} + \varepsilon), & ln^{-2} \leqslant x \leqslant ln^{-2} + \varepsilon, \\ n, & ln^{-2} + \varepsilon \leqslant x \leqslant ln^{-2} + n^{-3}, \\ n(1 - S(x, k; ln^{-2} + n^{-3}, ln^{-2} + n^{-3} + \varepsilon)), & \\ \qquad ln^{-2} + n^{-3} \leqslant x \leqslant ln^{-2} + n^{-3} + \varepsilon, & \\ 0, & ln^{-2} + n^{-3} + \varepsilon \leqslant x \leqslant (l+1) n^{-2}, \end{cases}$$

where $l = 0, 1, 2, ..., n^2 - 1$, and ε is very small ($\varepsilon \leqslant n^{-5}$ will do). The function g is $(k-1)$-times continuously differentiable, and, since $S(x) \in \Pi_{2k+1}$, then by Markov's inequality,

$$\|g^{(k-1)}\|_\infty \leqslant C(k) \, n\varepsilon^{1-k}.$$

Now, let a function f be such that $f(x) = bx^k + \int_0^x g(t) \, dt$, where b is chosen so that $f \in \Delta^k[0, 1]$. Of course, it is always possible since $f^{(k)}(x) = k! \, b + g^{(k-1)}(x) \geqslant k! \, b - C(k) \, n\varepsilon^{1-k} \geqslant 0$ for sufficiently large b. We now show how the rest of the proof can be reduced to Ditzian's construction in [3]. Let

$$G(x) = \begin{cases} n, & ln^{-2} < x < ln^{-2} + n^{-3}, \\ 0, & ln^{-2} + n^{-3} < x < (l+1) n^{-2}). \end{cases}$$

It was shown in [3] that

$$\left\| \int_0^x G(t)\,dt - x \right\|_p \leqslant Cn^{-2}.$$

Also,

$$\left\| \int_0^x (G(t) - g(t))\,dt \right\|_\infty \leqslant \int_0^1 |G(t) - g(t)|\,dt \leqslant 2\varepsilon n^3.$$

Therefore,

$$\begin{aligned}\left\| \int_0^x g(t)\,dt - x \right\|_p^p &\leqslant \left\| \int_0^x G(t)\,dt - x \right\|_p^p + \left\| \int_0^x (g(t) - G(t))\,dt \right\|_p^p \\ &\leqslant Cn^{-2p} + \left\| \int_0^x (g(t) - G(t))\,dt \right\|_\infty^p \\ &\leqslant Cn^{-2p} + C(\varepsilon n^3)^p \leqslant Cn^{-2p}.\end{aligned}$$

This implies that

$$\begin{aligned}\omega^{k+1}(f, n^{-1})_p^p &= \omega^{k+1}\left(\int_0^x g(t)\,dt - x, n^{-1} \right)_p^p \\ &\leqslant C \left\| \int_0^x g(t)\,dt - x \right\|_p^p \leqslant Cn^{-2p}.\end{aligned}$$

Also,

$$\omega^k(f', n^{-1})_p^p = \omega^k(g, n^{-1})_p^p \leqslant C\,\|g\|_p^p \leqslant Cn^{p-1}.$$

Suppose now that there exists a polynomial $P_n \in \mathbf{P}_n$ such that

$$\|f - P_n\|_p \leqslant C\omega^{k+1}(f, n^{-1})_p \leqslant Cn^{-2}$$

and

$$\|f' - P_n'\|_p \leqslant C\omega^k(f', n^{-1})_p \leqslant Cn^{1-1/p}.$$

Using the same sequence of inequalities as in [3] a contradiction can be obtained. We refer the reader to [3] for more details. ∎

4. RATE OF APPROXIMATION OF k-MONOTONE FUNCTIONS

Proof of Theorem 3. We use the idea from [8]. Note that it is sufficient to prove Theorem 3 only for $p=1$ and $p=\infty$, i.e.,

$$\|f-p_n\|_\infty \leqslant C\omega_\varphi^k(f, n^{-1})_\infty \tag{15}$$

and

$$\|f-p_n\|_1 \leqslant Cn^{-k}\,\|f^{(k-2)}\|_\infty. \tag{16}$$

Indeed, suppose that there exists a polynomial $p_n \in \mathbf{P}_n$ which satisfies (15) and (16). Then

$$\begin{aligned}\|f-p_n\|_p^p &= \int_{-1}^1 |f-p_n|^p = \int_{-1}^1 |f-p_n|^{p-1}\,|f-p_n| \\ &\leqslant \|f-p_n\|_\infty^{p-1}\,\|f-p_n\|_1,\end{aligned}$$

and, hence,

$$\|f-p_n\|_p \leqslant \|f-p_n\|_\infty^{1/q}\,\|f-p_n\|_1^{1/p} \leqslant Cn^{-k/p}\omega_\varphi^k(f, n^{-1})_\infty^{1/q}\,\|f^{(k-2)}\|_\infty^{1/p}.$$

Note that it is relatively easy to prove (15) and (16) without the requirement that the same polynomial p_n is used in both of them. When this requirement is present Theorem 1 becomes useful.

Lemma 9. *Let $f \in \Delta^2[-1, 1]$. Then*

$$\omega_\varphi^2(f, \delta)_1 \leqslant C\delta^2\,\|f\|_\infty, \tag{17}$$

where C is an absolute constant.

Proof. Using the definition of the second modulus of smoothness, keeping in mind that $\Delta_h^2 f(x) \geqslant 0$ for $f \in \Delta^2$ and changing variables, we have

$$\begin{aligned}&\int_{-1}^1 |\Delta_{h\varphi(x)}^2 f(x)|\,dx \\ &\quad = \int_{x\,:\,x \pm h\varphi(x) \in [-1, 1]} (f(x-h\varphi(x)) - 2f(x) + f(x+h\varphi(x)))\,dx\end{aligned}$$

$$
\begin{aligned}
&= \int_{-1}^{1-3h^2/1+h^2} f(y)(1+h^2)^{-1}\left(1-\frac{yh}{\sqrt{h^2+1-y^2}}\right) dy \\
&\quad -2\int_{-1+h^2/1+h^2}^{1-h^2/1+h^2} f(y)\, dy \\
&\quad +\int_{-1+3h^2/1+h^2}^{1} f(y)(1+h^2)^{-1}\left(1+\frac{yh}{\sqrt{h^2+1-y^2}}\right) dy \\
&\leqslant Ch^2\,\|f\|_\infty + \int_{-1+3h^2/1+h^2}^{1-3h^2/1+h^2} 2h^2(1+h^2)^{-1}\,|f(y)|\, dy \\
&\leqslant Ch^2\,\|f\|_\infty .
\end{aligned}
$$

This immediately implies (17). ∎

It only remains to notice that, for $p=\infty$, Theorem 2 ($j=0$) yields (15), and if $p=1$, then together with Lemma 9 it implies (16), since

$$\|f-p_n\|_1 \leqslant C\omega_\varphi^k(f, n^{-1})_1 \leqslant Cn^{-k+2}\omega_\varphi^2(f^{(k-2)}, n^{-1})_1 \leqslant Cn^{-k}\,\|f^{(k-2)}\|_\infty,$$

since $f^{(k-2)} \in \Delta^2[-1, 1]$. This completes the proof of Theorem 3. ∎

REFERENCES

1. P. S. Bullen, A criterion for n-convexity, *Pacific J. Math.* **36** (1971), 81–98.
2. Z. Ciesielski, Some properties of convex functions of higher orders, *Ann. Polon. Math.* **7** (1959), 1–7.
3. Z. Ditzian, A note on simultaneous polynomial approximation in $L_p[-1, 1]$, $0<p<1$, *J. Approx. Theory* **82** (1995), 317–319.
4. K. G. Ivanov, Approximation to convex functions by means of polynomials and polygons in L_1-metric, *in* "Approximation and Function Spaces (Gdańsk, 1979)," pp. 287–293, North-Holland, Amsterdam/New York, 1981.
5. K. A. Kopotun, A note on simultaneous polynomial approximation in $L_p[-1, 1]$, $(1 \leqslant p < \infty)$, *Analysis* **15** (1995), 151–158.
6. K. Kopotun, Simultaneous approximation by algebraic polynomials, *Constr. Approx.* **12** (1996), 67–94.
7. M. Nikoltjeva-Hedberg, Approximation of a convex function by convex algebraic polynomials in L_p, $1 \leqslant p < \infty$, *J. Approx. Theory* **73** (1993), 288–302.
8. M. Nikoltjeva-Hedberg and V. Operstein, A note on convex approximation in L_p, *J. Approx. Theory* **81** (1995), 141–144.
9. J. E. Pečarić, F. Proschan, and Y. L. Tong, "Convex Functions, Partial Orderings, and Statistical Applications," Mathematics in Science and Engineering, Vol. 187, Academic Press, Boston, 1992.
10. A. W. Roberts and D. E. Varberg, "Convex Functions," Academic Press, New York, 1973.
11. M. P. Stojanova, Approximation of a convex function by algebraic polynomials in $\mathbf{L}_p[a, b]$ $(1<p<\infty)$, *Serdica* **11** (1985), 392–397.